Math Circles - Pigeonhole Principle - Fall 2022

Exercises

1. Suppose that S is a set of $n+1$ integers. Prove that S contains distinct integers a and b such that $b-a$ is a multiple of n.

2. Let S be a set of 10 distinct integers between 1 and 60, inclusive. Prove that we can choose two disjoint[1] subsets of S (say, S_1 and S_2) such that the sum of the elements in S_1 is equal to the sum of the elements in S_2.

3. Show that in any set of 100 integers, one can choose 15 of them such that the difference between any two is divisible by 7.

4. Prove that in any set of 100 integers, one can choose a set of at least one number whose sum is divisible by 100.

5. Suppose that the numbers $0, 1, 2, \ldots, 9$ are randomly assigned to the vertices of a decagon.[2] Show that there are three consecutive vertices whose sum is at least 14.

6. Let S be a set of 3 distinct integers. Show that one can always choose two of them (say, a and b) such that $ab(a-b)(a+b)$ is divisible by 10.

7. (HARD)
   Show that any positive integer x containing N digits, none of which are 0, is either divisible by N or can be converted into an integer that is divisible by N by replacing some, but not all, of its digits with 0.

[1] *Disjoint* means that the sets have no elements in common; that is, if x is in S_1 then x is not in S_2.
[2] A *decagon* is a polygon with 10 vertices.